V 03.04.2018. (VIII nedelja)

Granični zadatak za l.d.j.

pr. $x'' + x = 0$, $t \in (0,1)$

a) $x'(0) = 0$, $x'(1) = 1$ ima jedinstveno rešenje

b) $x'(0) = 0$, $x'(\pi) = 1$ nema rešenja

c) $x'(0) = 0$, $x'(\pi) = 0$ beskonačno rešenja

gr. uslovi
$$\begin{cases} x^{(n)} + a_1(t)x^{(n-1)} + \dots + a_n(t)x = b(t) \quad \text{tj. } L(x) = b(t) \\ u_1(x) = M_{11}x(\alpha) + \dots + M_{1n}x^{(n-1)}(\alpha) + N_{11}x(\beta) + \dots + N_{nn}x^{(n-1)}(\beta) = \gamma_1 \\ \vdots \\ u_n(x) = M_{n1}x(\alpha) + \dots + M_{nn}x^{(n-1)}(\alpha) + N_{nn}x(\beta) + \dots + N_{nn}x^{(n-1)}(\beta) = \gamma_n \end{cases}$$

$$u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \quad \gamma = \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{pmatrix}, \quad u(x) = \gamma$$

$$\begin{cases} L(x) = b(t) \\ u(x) = \gamma \end{cases}$$

★ Prvo se bavimo ovim zadatkom:

$$\begin{cases} L(x) = 0 \\ u(x) = 0 \end{cases}, \quad x \equiv 0 \text{ uvijek rešenje}$$

Kada imamo netrivijalno rešenje:

$x = c_1\Phi_1 + \dots + c_n\Phi_n$

$$u_j\left(\sum_{i=1}^{n} c_i\phi_i\right) = 0, \quad j = \overline{1,n}$$

$$\sum_{i=1}^{n} c_i u_j(\Phi_i) = 0, \quad j = \overline{1,n}$$

$$\Delta = \begin{vmatrix} u_1(\Phi_1) & \dots & u_1(\Phi_n) \\ \vdots & & \vdots \\ u_n(\Phi_1) & \dots & u_n(\Phi_n) \end{vmatrix} = 0 \quad \text{(tada ima netriv. reš.)}$$

Lema: Granični zadatak ★ ima netrivijalno reš. akko $\Delta = 0$.

$$\begin{cases} L(x) = b(t) \\ u(x) = \gamma \end{cases}$$

$x = y + v$ t.d. $u(v) = \gamma \Rightarrow u(x) = \gamma \Rightarrow u(y) + u(v) = \gamma \Rightarrow u(y) = 0$

$L(x) = b(t)$

$L(y) + L(u) = b(t) \Rightarrow L(y) = b(t) - L(u)$

$$\begin{cases} L(y) = c(t) \\ u(y) = 0 \end{cases}$$

Teorema: Ako homogeni granični zadatak ★ ima samo trivijalno rešenje, tada odgovarajući nehomogeni granični zadatak ima jedinstveno rešenje.

$$x = \int_\alpha^\beta G(t,s)\, b(s)\, ds \quad \text{- Grinova f-ja}$$

za $n = 2$: $x'' + a_1(t)x' + a_2(t)x = b(t)$ (2) zadatak

$$\left.\begin{aligned} &\alpha_0 x'(\alpha) + \alpha_1 x(\alpha) = 0 \quad (3) \\ &\beta_0 x'(\beta) + \beta_1 x(\beta) = 0 \quad (4) \end{aligned}\right\} \text{uslovi}$$

Nađemo x_1 - rešenje za pripadnu homogenu j-nu (2) koje zadovoljava (3) i x_2 - rešenje -//- koje zadovoljava (4).

Tada:

$$w(s) = \begin{vmatrix} x_1(s) & x_2(s) \\ x_1'(s) & x_2'(s) \end{vmatrix}$$

$$G(t,s) = \begin{cases} \dfrac{x_1(s)\, x_2(t)}{w(s)}, & \alpha \le s \le t \\[2ex] \dfrac{x_1(t)\, x_2(s)}{w(s)}, & t \le s \le \beta \end{cases}$$

pr. $x'' - x = 2t$

$x(0) = 0 \quad , \quad x(1) = -1$

R: karak. j-na: $\lambda^2 - 1 = 0 \iff \lambda^2 = 1 \Rightarrow \lambda = \pm 1$

$\Rightarrow x_h = c_1 e^t + c_2 e^{-t}$

$x_p = At + B \Rightarrow x_p = -2t$

$\Rightarrow x = x_h + x_p = c_1 \cdot e^t + c_2 e^{-t} - 2t$

$$\left.\begin{array}{l} x(0) = 0 \Rightarrow c_1 + c_2 = 0 \Rightarrow c_1 = -c_2 \quad (1) \\ x(1) = -1 \Rightarrow c_1 \cdot e^1 + c_2 e^{-1} - 2 = -1 \\ \qquad \Rightarrow c_1 \left(e - \frac{1}{e}\right) = 1 \quad (2) \end{array}\right\} \Rightarrow \begin{array}{l} c_1 = \frac{e}{e^2 - 1} \\ c_2 = \frac{-e}{e^2 - 1} \end{array}$$

$x'' = f(t)$

$x(0) = 0 \qquad \Rightarrow \alpha = 0, \beta = 1$

$x(1) = 0$

R: $x'' = 0 \qquad / \iint$ ili $\lambda^2 = 0 \quad r = 2$

$x = At + B$

Tražimo jedno rešenje koje treba da zadovoljava 1°

$x_1(0) = 0 \Rightarrow A \cdot 0 + B = 0 \Rightarrow B = 0$

$\boxed{x_1 = t}$ (uzmimo $A = 1$)

— // — zadovoljava 2°

$x_2(1) = 0 \Rightarrow A \cdot 1 + B = 0 \Rightarrow A = -B$ (uzmimo $A = 1, B = -1$)

$\boxed{x_2 = t - 1}$

$$\underset{w(t)}{\overset{\shortparallel}{w(s)}} = \begin{vmatrix} t & t-1 \\ 1 & 1 \end{vmatrix} = t - (t-1) = 1$$

$$G(s,t) = \begin{cases} \frac{s(t-1)}{1}, & 0 \le s \le t \\ \frac{t(s-1)}{1}, & t \le s \le 1 \end{cases}$$

npr. za $f(t) = t$

$$x(t) = \int_0^1 G(s,t)\, f(s)\, ds$$

$$x(t) = \int_0^1 G(s,t)\, s\, ds = \int_0^t s(t-1)\, s\, ds + \int_t^1 t(s-1)\, s\, ds =$$

$$= (t-1)\frac{t^3}{3} + t\left(\frac{s^3}{3}\Big|_1^t - \frac{s^2}{2}\Big|_1^t\right) = \frac{t^3}{6} - \frac{t}{6}$$

$x'' = t$

$\left(\frac{t^3}{6} - \frac{t}{6}\right)'' = t$ ← tako znamo da je sigurno rešenje

// Komentar: $x'' = 0$, $x(0) = 0$

$x(1) = 0$

$x = At + B$

$x(0) = 0 \Rightarrow B = 0 \Rightarrow x = At$

$x(1) = 0 \Rightarrow A = 0$

homogena ima samo jedinstveno reš. 0 pa znamo da možemo koristiti Grina

2. $x'' + x = f(t)$

$x'(0) = 0 \quad \Rightarrow \alpha = 0,\ \beta = \pi$

$x(\pi) = 0$

R:

$x'' + x = 0 \Leftrightarrow \lambda^2 + 1 = 0 \Leftrightarrow \lambda^2 = -1 \Leftrightarrow \lambda = \pm i$

$x_h = c_1 \cos t + c_2 \sin t$

x_1

$x'(0) = 0 \Rightarrow -c_1 \cancel{\sin 0} + c_2 \cos 0 = 0 \Rightarrow c_2 = 0$

uzmimo $c_1 = 1 \Rightarrow x_1 = \cos t$

x_2

$x(\pi) = 0 \Rightarrow c_1 \cos\pi + c_2 \cancel{\sin\pi} = 0 \Rightarrow c_1 = 0$

uzmimo $c_2 = 1 \Rightarrow x_2 = \sin t$

$$w(s) = \begin{vmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{vmatrix} = 1$$

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$$G(t,s) = \begin{cases} \cos s \cdot \sin t & , \ 0 \le s \le t \\ \cos t \ \sin s & , \ t \le s \le \pi \end{cases}$$

npr. za $f(t) = 1$

$$x(t) = \int_0^t \cos s \cdot \sin t \cdot 1 \, ds + \int_t^{\pi} \cos t \cdot \sin s \cdot 1 \, ds =$$

$$= \sin^2 t - \cos\pi \cos t + \cos^2 t = 1 + \cos t$$

$$x'' - x = f(t) \qquad \Rightarrow \alpha = 0, \ \beta = 2$$

$$x'(0) = 0; \ x'(2) + x(2) = 0$$

$$x'' - x = 0 \Leftrightarrow \lambda^2 - 1 = 0 \Leftrightarrow \lambda^2 = 1 \Rightarrow \lambda = \pm 1$$

$$x_h = c_1 e^t + c_2 e^{-t}$$

$\underline{x_1}$

$$x'(0) = 0 \Rightarrow c_1 e^0 - c_2 e^0 = 0 \Rightarrow c_1 - c_2 = 0 \Rightarrow c_1 = c_2$$

uzmimo $c_1 = c_2 = 1 \Rightarrow x_1 = e^t + e^{-t}$

$\underline{x_2}$

$$x'(2) + x(2) = 0 \Rightarrow c_1 e^2 - \cancel{c_2 e^{-2}} + c_1 e^2 + \cancel{c_2 e^{-2}} = 0$$

$$2c_1 e^2 = 0 \Rightarrow c_1 = 0$$

uzmimo $c_2 = 1 \Rightarrow x_2 = e^{-t}$

$$w(s) = \begin{vmatrix} e^s + e^{-s} & e^{-s} \\ e^s - e^{-s} & -e^{-s} \end{vmatrix} = -1 - e^{-2s} - e^{-s}(e^s - e^{-s}) = -1 - \cancel{e^{-2s}} - 1 + \cancel{e^{-2s}} = -2$$

$$G(t,s) = \begin{cases} \dfrac{(e^s + e^{-s}) e^{-t}}{-2} & , \ 0 \le s \le t \\ \dfrac{(e^t + e^{-t}) e^{-s}}{-2} & , \ t \le s \le 2 \end{cases}$$

npr. za $f = e^t$

$$x(t) = \int_0^t \frac{(e^s + e^{-s}) \cdot e^s \cdot e^{-t}}{-2} ds + \int_t^2 \frac{(e^t + e^{-t}) \cdot e^{-s} \cdot e^s}{-2} ds =$$

$$= \dots =$$

$$= -\frac{5}{4} e^t - \frac{3}{4} e^{-t} + \frac{t}{2} e^t$$

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Za vježbu: $x'' + x = f(t)$

$x(0) = x'(0)$

$x(\pi) = 2x'(\pi)$

4. $t x'' - x' = f(t)$

$x'(1) = 0$

$x(2) = 0$

I način: $x' = z \Rightarrow x'' = z'$

$t z' - z = 0$

$t z' = z$

$\frac{z'}{z} = \frac{1}{t} \int dt \quad , z \neq 0$

$\ln|z| = \ln|t| + \ln|C_1|$

$z = C_1 t$

$x' = C_1 t$

$x_h = A t^2 + B$

II način: $t x'' - x' = 0 \quad / \cdot t$

$t^2 x'' - t x' = 0$ Ojlerova

$x = t^\lambda$

$t^\lambda (\lambda(\lambda - 1) - \lambda) = 0$

$\lambda^2 - 2\lambda = 0$

$\lambda(\lambda - 2) = 0$

$\lambda = 0 \quad \vee \quad \lambda = 2$

$x_h = A t^2 + B$

x_1: $x'(1) = 0 \Rightarrow A = 0$

uzmimo $B = 1 \Rightarrow x_1 = 1$

x_2: $x(2) = 0 \Rightarrow 4A + B = 0 \Rightarrow B = -4A$

$(A, B) = (1, -4)$

$x_2 = t^2 - 4$

Pripadni homogeni zadatak ima samo trivijalno rj.

$$\omega(s) = \begin{vmatrix} 1 & s^2 - 4 \\ 0 & 2s \end{vmatrix} = 2s$$

$$G(s,t) = \begin{cases} \frac{t^2 - 4}{2s}, & 1 \le s \le t \\ \frac{s^2 - 4}{2s}, & t \le s \le 2 \end{cases}$$

Za $f(t) = t^3$

$tx'' - x' = t^3$

$x'' - \frac{1}{t}x' = t^2$

$$x(t) = \int_1^t \frac{t^2-4}{2s} \cdot s^2\, ds + \int_t^2 \frac{s^2-4}{2s} \cdot s^2\, ds = \dots = \frac{t^4}{8} - \frac{1}{4}t^2 - 1$$

$x''(t) = \frac{12}{8}t^2 - \frac{2}{4}$

$tx'' - x' = t\left(\frac{12}{8}t^2 - \frac{2}{4}\right) - \left(\frac{4t^3}{8} - \frac{2}{4}t\right) = t^3$

$t^2x'' + 2tx' = f(t)$

$x(1) = 0 \qquad \Rightarrow \alpha = 1, \beta = 3$

$x'(3) = 0$

R

$t^2x'' + 2tx' = 0$ Ojlerova DJ

$x = t^\lambda$ ili $z = \ln(t)$.

$\lambda(\lambda-1) + 2\lambda = 0 \Rightarrow \lambda^2 + \lambda = 0 \Rightarrow \lambda(\lambda+1) = 0$ $\nearrow \lambda = 0$, $\searrow \lambda = -1$

$x_h = c_1 + \frac{c_2}{t}$

x_1: $x(1) = 0 \Rightarrow c_1 + c_2 = 0 \Rightarrow c_1 = -c_2$

uzmimo $c_1 = 1, c_2 = -1 \Rightarrow x_1 = 1 - \frac{1}{t}$

x_2: $\frac{-c_2}{9} = 0 \Rightarrow c_2 = 0$

uzmimo $c_1 = 1 \Rightarrow x_2 = 1$

Pripadni homogeni ima jedinstveno rješ.

$$W = \begin{vmatrix} 1-\frac{1}{s} & 1 \\ \frac{1}{s^2} & 0 \end{vmatrix} = -\frac{1}{s^2}$$

$$G(s,t) = \begin{cases} \dfrac{(1-\frac{1}{s})\,1}{-\frac{1}{s^2}}, & 1 \le s \le t \\ \dfrac{(1-\frac{1}{t})}{-\frac{1}{s^2}}, & s \le t \le 3 \end{cases}$$

$f(t) = t^2$

$$x(t) = \int_1^3 G(s,t)\, b(s)\, ds = \_$$

t^{-2}

- $t^2 x'' - t x' - x = f(t)$ – Ojlerova

$x(1) = 0$, $x(t)$ je ograničena f-ja $t \to \infty$

$\underline{\underline{R}}$

$t^{\alpha}, \ldots, \alpha = \pm 1$

$x = c_1 t + \frac{c_2}{t} \Rightarrow x_2 = \frac{1}{t}, \quad x_1 = t - \frac{1}{t}$

- $x'' - x = f(t)$

$x(t)$ ograničeno za $t \to \pm\infty$

$\underline{\underline{R}}$

$\lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$

$x = c_1 e^t + c_2 \cdot e^{-t}$

tako će biti ogr.

$\underline{\underline{x_1}}$ $t \to \infty \Rightarrow e^{-t} \to 0$, uzmimo $c_1 = 0 \Rightarrow x_1 = e^{-t}$ $(c_2 = 1)$

$\underline{\underline{x_2}}$ $t \to -\infty \Rightarrow e^{t} \to 0$, uzmimo $c_2 = 0 \Rightarrow x_2 = e^{t}$ $(c_1 = 1)$

$$w(s) = \begin{vmatrix} e^s & e^{-s} \\ e^s & -e^{-s} \end{vmatrix} = -2$$

~~$x'' - t x'$~~

$$G(t,s) = \begin{cases} \dfrac{e^s e^{-t}}{-2}, & -\infty \le s \le t \\[2ex] \dfrac{e^t e^{-s}}{-2}, & t \le s \le +\infty \end{cases}$$

Granični zadatak sa parametrom

$$\begin{cases} \boxed{x'' - \lambda x = 0} \\ x(0) = 0 \\ x(\ell) = 0, \quad \ell > 0 \end{cases}$$

– homogena, pa ne mora Grinova

1) $\lambda > 0$ $\qquad \xi^2 - \lambda = 0$

$\xi = \pm\sqrt{\lambda}$

$x = c_1 e^{\sqrt{\lambda} t} + c_2 e^{-\sqrt{\lambda} t}$

$x(0) = 0 \Rightarrow c_1 + c_2 = 0 \Rightarrow c_1 = -c_2$

$x(\ell) = 0 \Rightarrow c_1 e^{\sqrt{\lambda}\ell} - c_1 e^{-\sqrt{\lambda}\ell} = 0 \Rightarrow c_1 (e^{\sqrt{\lambda}\ell} - e^{-\sqrt{\lambda}\ell}) = 0$

$c_1=0 \Rightarrow c_2=0\,,\ x\equiv 0$

$e^{\sqrt{\lambda}\,l} = e^{-\sqrt{\lambda}\,l}$ ↯

2) $\lambda=0$

$x = At+B$

$x(0)=0 \Rightarrow B=0\,,\ x=At$

$x(l)=0 \Rightarrow A=0\ ;\ x\equiv 0$

3) $\lambda<0$

$\xi^2=\lambda \Rightarrow \xi=\pm i\sqrt{-\lambda}$

$x = c_1\cos(\sqrt{-\lambda}\,t)+c_2\sin(\sqrt{-\lambda}\,t)$

$x(0)=0 \Rightarrow c_1=0$

$x = c_2\sin(\sqrt{-\lambda}\,t)$

$x(l)=0 \Rightarrow \sin(\sqrt{-\lambda}\,l)=0$

Kada ima smisla da tražimo netrivijalna rešenja?

Treba da važi $\sqrt{-\lambda_k}\,l = k\pi,\ k\in\mathbb{Z}$

$\sqrt{-\lambda_k} = \frac{k\pi}{l} \quad /^2$

$-\lambda_k = \frac{k^2\pi^2}{l^2} \Rightarrow \lambda_k = -\frac{k^2\pi^2}{l^2} \Rightarrow \boxed{X_k = C\sin\left(\frac{k\pi}{l}t\right)}$

Za vježbu: $x''+4x'+3x=f(t)$

$x(0)=0 \qquad \alpha=0$

$x(t)=o(e^{-2t}),\ t\to+\infty \qquad \beta=+\infty$

$\underline{\underline{2}}$

$\lambda^2+4\lambda+3=0 \Rightarrow \lambda_{1/2}=\frac{-4\pm\sqrt{16-12}}{2}=\frac{-4\pm 2}{2} \Rightarrow \lambda_1=-1,\ \Rightarrow \lambda_2=-3$

$x_h = c_1e^{-t}+c_2e^{-3t}$

$\underline{\underline{x_1}}$ $c_1+c_2=0 \Rightarrow c_1=-c_2 \Rightarrow x_1=e^{-t}-e^{-3t}$ za $c_1=1,\ c_2=-1$

$\underline{\underline{x_2}}$ $x_2=e^{-3t}$ jer $c_1e^{-t}+c_2e^{-3t}$ $\qquad X_2=e^{-3t}$

$\frac{X_1}{e^{-2t}} = c_1e^{t}+c_2e^{-t} \Rightarrow c_1=0,\quad c_2=1$

(Zad) Koristeći Greenovu funkciju riješiti graničnu zadaću

$t^2x'' - (4t+t^2)x' + x(6+2t) = t^2+2t+6$

$2x(1) - x'(1) = 2 \; ; \; 5x(3) - 3x'(3) = 5.$

(1) Pripadna homogena

$$\underbrace{t^2x''}_{m} - \underbrace{(4t+t^2)\,x'}_{m+1} + \underbrace{x(6+2t)}_{m+1} = 0$$

$x = t^m + \dots$

$t^{m+1}(-m+2) + \dots = 0.$

$-m+2=0$

$m=2$

Tražimo u obliku
$x = t^2 + bt + c.$
$x' = 2t + b$
$x'' = 2$

$t^2 2 - (4t+t^2)(2t+b) + (t^2+bt+c)(6+2t) = 0.$

$t^3(-2+2) + t^2(2-8-b+6+2b)$
$+ t(-4b+6b+2c) + 6c = 0$

$$\begin{cases} b=0 \\ 2b+2c=0 \\ 6c=0. \end{cases}$$

$\Rightarrow 6c=0 \Rightarrow c=0 \Rightarrow b=0.$

$\underline{x=t^2}$

$x = t^2 \cdot y$

$x' = 2ty + t^2y'$

$x'' = 2y + 4ty' + t^2y''$

$t^2(2y+4ty'+t^2y'') - (4t+t^2)(2ty+t^2y') + t^2y(6+2t) = 0$

$y''(t^4) + y'(\cancel{4t^3} - \cancel{4t^3} - t^4) + y(\cancel{2t^2} - \cancel{8t^2} - \cancel{2t^3} + \cancel{6t^2} + \cancel{2t^3}) = 0$

$y''t^4 + y'(-t^4) = 0 \Rightarrow y'' - y' = 0.$

$y'' - y' = 0 \qquad \lambda^2 - \lambda = 0 \qquad y = c_1 e^t + c_2$

$\lambda = 1, \; \lambda = 0$

$x = t^2 (c_1 e^t + c_2) = \underline{c_1 t^2 e^t + c_2 t^2}$ → Opšte homogene

$v = At + B$

$v(1) = A + B \qquad v(3) = 3A + B$

$v'(1) = A \qquad v'(3) = A$

$2v(1) - v'(1) = 2 \qquad 5v(3) - 3v'(3) = 5$

$2(A+B) - A = 2 \qquad 5(3A+B) - 3A = 5$

$A + 2B = 2 \qquad 12A + 5B = 5$

$\boxed{\begin{matrix} A = 0 \\ B = 1 \end{matrix}}$

Smjena $\quad x = y + 1 \qquad x' = y' \quad x'' = y''$

$t^2 y'' - (4t + t^2) y' + y(6 + 2t) + (6 + 2t) = t^2 + 2t + 6$

$\boxed{t^2 y'' - (4t + t^2) y' + y(6 + 2t) = t^2}$

$2x(1) - x'(1) = 2 \qquad 5(y(3)+1) - 3y'(3) = 5$

$2(y(1)+1) - y'(1) = 2 \qquad 2^\circ \; \boxed{5y(3) - 3y'(3) = 0}$

$1^\circ \; \boxed{2y(1) - y'(1) = 0}$

$\Rightarrow y_h = c_1 t^2 e^t + c_2 t^2 \qquad y_h' = c_1(2te^t + t^2 e^t) + 2c_2 t$

$1^\circ \quad 2(c_1 e + c_2) - (c_1(2e + e) + 2c_2) = 0$

$c_1 = 0$

Recimo $c_2 = 1 \qquad \boxed{y_1 = t^2}$

$2^\circ \quad 5(9c_1 e^3 + 9c_2) - 3(c_1(6e^3 + 9e^3) + 6c_2) = 0$

$c_1(\underbrace{45e^3 - 45e^3}_{0}) + 45c_2 - 18c_2 = 0$

$\Rightarrow c_2 = 0 \qquad \Rightarrow c_1 = 1$

$\boxed{y_2 = t^2 e^t}$

a' $W(s) = \begin{vmatrix} s^2 & s^2e^s \\ 2s & 2se^s+s^2e^s \end{vmatrix} = \cancel{2s^3e^s} + s^4e^s - \cancel{2s^3e^s}$ 3

$= s^4e^s.$

$$G(t,s) = \begin{cases} \dfrac{s^2t^2e^t}{s^4e^s}, & 1 \le s \le t \\[2ex] \dfrac{t^2s^2e^s}{s^4e^s}, & t \le s \le 3. \end{cases}$$

$b(s) = 1$

$$y(t) = \int_1^3 G(t,s)\,b(s)\,ds = \int_1^t \frac{1}{s^2e^s}\,t^2e^t\,ds + \int_t^3 \frac{t^2}{s^2}\,ds = \ldots$$

$$\underline{\underline{x(t) = y(t) + 1}}$$